

## TRANSLATION SURFACES OF LINEAR WEINGARTEN TYPE

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ABSTRACT. We give a relatively simple proof that a translation surface in Euclidean space that satisfies a relation of type  $aH + bK = c$ , for some real numbers  $a, b, c$ , where  $H$  and  $K$  are the mean curvature and the Gauss curvature of the surface, respectively, must have  $a = 0$  or  $b = 0$ , and thus,  $K$  is constant or  $H$  is constant. Our method of proof extends to the Lorentzian ambient space.

## 1. INTRODUCTION AND RESULTS.

A Weingarten surface in Euclidean space  $\mathbb{R}^3$  is a surface  $S$  whose mean curvature  $H$  and Gauss curvature  $K$  satisfies a non-trivial relation  $\Psi(H, K) = 0$ . This type of surfaces were introduced by the very Weingarten in the context of the problem of finding all surfaces isometric to a given surface of revolution and have been extensively studied in the literature [13]. In order to simplify the study of Weingarten surfaces, it is natural to impose some added geometric condition on the surface, as for example, that  $S$  is ruled or rotational [1, 3, 4, 7, 12].

Following this strategy, Dillen, Goemans and Van de Woestyne considered Weingarten surfaces that are graphs of type  $z = f(x) + g(y)$ , where  $f$  and  $g$  are smooth functions defined in some intervals  $I, J \subset \mathbb{R}$ , respectively [2]. A surface  $S$  in  $\mathbb{R}^3$  is called a *translation surface* if it can locally parametrize as  $X(x, y) = (x, y, f(x) + g(y))$ . In particular, a translation surface  $S$  has the property that the translations of a parametric curve  $x = ct$  by the parametric curves  $y = ct$  remain in  $S$  (similarly for the parametric curves  $x = ct$ ). In the cited paper, the authors classify all translation surfaces of Weingarten type:

**Theorem A** ([2]). *A translation surface in  $\mathbb{R}^3$  of Weingarten type is a plane, a generalized cylinder, a Scherk's minimal surface or an elliptic paraboloid.*

The proof given in [2] (see also [6]) discusses many cases and it involves the solvability of a large number of ODE systems. In fact, in [2] it is described the procedure and it requires of calculations which are done with a computer program (as Maple) to manipulate the algebraic operations. This is the reason that some authors previously obtained partial results assuming simpler functions  $f$  and  $g$ , as for example, that they are polynomial in its variables, simplifying and doing easier the computations ([11, 15]).

In this paper we provide a significantly simpler proof of Th. A when the Weingarten relation is linear in its variables. A *linear Weingarten surface* in Euclidean

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2000 *Mathematics Subject Classification.* 53A05, 53A10, 53A35.

*Key words and phrases.* translation surface, linear Weingarten surface, mean curvature, Gauss curvature.

Partially supported by MEC-FEDER grant no. MTM2011-22547 and Junta de Andalucía grant no. P09-FQM-5088.

space  $\mathbb{R}^3$  is a surface where there exists a relation

$$(1) \quad a H + b K = c,$$

for some real numbers  $a, b, c$ , not all zero. In the class of linear Weingarten surfaces, we mention two families of surfaces that correspond with trivial choices of the constants  $a$  and  $b$ : surfaces with constant Gauss curvature ( $a = 0$ ) and surfaces with constant mean curvature ( $b = 0$ ). In Th. A, only the three first surfaces are linear Weingarten surfaces, which have constant  $H$  or constant  $K$ : a plane ( $H = K = 0$ ), a generalized cylinder ( $K = 0$ ) and the Scherk's minimal surface parametrized as  $z = \log(\cos(\lambda y)) - \log(\cos(\lambda x))$ ,  $\lambda > 0$  ( $H = 0$ ). Besides these two families of surfaces, the classification of linear Weingarten surfaces in the general case is almost completely open today. See [5, 9, 12].

The result that we prove is:

**Theorem 1.** *A translation surface in Euclidean space  $\mathbb{R}^3$  of linear Weingarten type is a surface with constant Gauss curvature  $K$  or constant mean curvature  $H$ . In particular, the surface is congruent with a plane, a generalized cylinder or a Scherk's minimal surface.*

This proves that in the family of translation surfaces, there doesn't exist new linear Weingarten surfaces besides the trivial choices of  $a, b$  in (1). We point out that an early work of Liu proved that the only translations surfaces with constant  $K$  or constant  $H$  are the three first surfaces of Th. 1 ([8]). Finally, and with minor modifications, we extend in Th. 2 our results to the Lorentzian ambient space (see also [2]).

## 2. PROOF OF THEOREM 1

The mean curvature  $H$  and the Gauss curvature  $K$  are expressed in a local parametrization  $X$  as

$$(2) \quad H = \frac{eG - 2fF + gE}{2(EG - F^2)}, \quad K = \frac{eg - f^2}{EG - F^2},$$

where  $\{E, F, G\}$  and  $\{e, f, g\}$  are the coefficients of the first fundamental form and the second fundamental form, respectively. Assume that  $S$  is a translation surface expressed locally as  $X(x, y) = (x, y, f(x) + g(y))$  for some smooth functions  $f$  and  $g$ . Then  $H$  and  $K$  are

$$(3) \quad H = \frac{f''(1 + g'^2) + g''(1 + f'^2)}{2(1 + f'^2 + g'^2)^{\frac{3}{2}}}, \quad K = \frac{f''g''}{(1 + f'^2 + g'^2)^2}.$$

Suppose now that  $S$  is also a linear Weingarten surface, where  $H$  and  $K$  satisfy the linear relation (1). The proof of Theorem 1 is by contradiction and we suppose that  $a, b \neq 0$ . Let us observe that this implies  $f'' \neq 0$  and  $g'' \neq 0$  because on the contrary, and from (3),  $H$  is constant. Let

$$W = EG - F^2 = 1 + f'^2 + g'^2.$$

We distinguish two cases according the value of  $c$ .

2.1. **Case  $c = 0$ .** Suppose  $c = 0$  in (1). With the change  $a \rightarrow 2a$  and by using (3), Equation (1) writes as

$$(4) \quad a \frac{f''(1 + g'^2) + g''(1 + f'^2)}{(1 + f'^2 + g'^2)^{\frac{3}{2}}} + b \frac{f''g''}{(1 + f'^2 + g'^2)^2} = 0.$$

We multiply (4) by  $W^2$  and divide by  $(1 + g'^2)(1 + f'^2)$  obtaining

$$(5) \quad a \left( \frac{f''}{1 + f'^2} + \frac{g''}{1 + g'^2} \right) \sqrt{W} + b \frac{f''}{1 + f'^2} \frac{g''}{1 + g'^2} = 0.$$

Introduce the next notation:

$$(6) \quad F = \frac{f''}{1 + f'^2}, \quad G = \frac{g''}{1 + g'^2}.$$

In particular, since  $f'' \neq 0$  and  $g'' \neq 0$ , then  $F \neq 0$  and  $G \neq 0$ . Then (5) writes as

$$(7) \quad a(F + G)\sqrt{W} + bFG = 0.$$

Let us observe that this identity implies  $F + G \neq 0$ , since on the contrary,  $b = 0$ . From (7), we have

$$1 + f'^2 + g'^2 = W = \frac{b^2}{a^2} \left( \frac{FG}{F + G} \right)^2.$$

We differentiate this equation with respect to  $x$  and next, with respect to  $y$ . Because the left hand side is a sum of a function of  $x$  and a function  $y$ , this calculation yields 0. On the other hand, the right hand side concludes

$$(8) \quad 6 \frac{b^2}{a^2} \frac{F^2 G^2 F' G'}{(F + G)^4} = 0.$$

This implies  $F' = G' = 0$  and thus,  $F$  and  $G$  are constants. From (7), we deduce that  $W = 1 + f'^2 + g'^2$  is constant, in particular,  $f'$  and  $g'$  are constant: a contradiction with the fact that  $f'', g'' \neq 0$ .

2.2. **Case  $c \neq 0$ .** Consider  $c \neq 0$  in (1). Dividing by  $c$ , and after a change of notation, the relation (1) writes as

$$(9) \quad a \frac{f''(1 + g'^2) + g''(1 + f'^2)}{(1 + f'^2 + g'^2)^{\frac{3}{2}}} + b \frac{f''g''}{(1 + f'^2 + g'^2)^2} = 1,$$

or equivalently

$$(10) \quad a(F + G)\sqrt{W} + bFG = \frac{W^2}{(1 + f'^2)(1 + g'^2)},$$

where  $F$  and  $G$  are given in (6). We differentiate (10) separately with respect to  $x$  and with respect to  $y$ :

$$\begin{aligned} a \left( F' \sqrt{W} + (F + G) \frac{f' f''}{\sqrt{W}} \right) + b F' G &= \frac{4W f' f''}{(1 + f'^2)(1 + g'^2)} - \frac{2f' f'' W^2}{(1 + f'^2)^2 (1 + g'^2)}. \\ a \left( G' \sqrt{W} + (F + G) \frac{g' g''}{\sqrt{W}} \right) + b F' G &= \frac{4W g' g''}{(1 + f'^2)(1 + g'^2)} - \frac{2g' g'' W^2}{(1 + f'^2)(1 + g'^2)^2}. \end{aligned}$$

Dividing the first equation by  $f' f''$  and the second one by  $g' g''$ , we have

$$a \frac{F' \sqrt{W}}{f' f''} + b \frac{F' G}{f' f''} + \frac{2W^2}{(1 + f'^2)^2 (1 + g'^2)} = a \frac{G' \sqrt{W}}{g' g''} + b \frac{F G'}{g' g''} + \frac{2W^2}{(1 + f'^2)(1 + g'^2)^2}.$$

From (10), we replace the value of  $W^2$  in the above expression, obtaining

$$(11) \quad \begin{aligned} & a \left( \frac{F'}{f'f''} + \frac{2(F+G)}{1+f'^2} - \frac{G'}{g'g''} - \frac{2(F+G)}{1+g'^2} \right) \sqrt{W} \\ & + b \left( \frac{F'G}{f'f''} + \frac{2FG}{1+f'^2} - \frac{FG'}{g'g''} - \frac{2FG}{1+g'^2} \right) = 0. \end{aligned}$$

Now we write (9) as

$$a(f''(1+g'^2) + g''(1+f'^2))\sqrt{W} + bf''g'' = W^2$$

and we differentiate this expression with respect to  $x$  and with respect to  $y$ :

$$\begin{aligned} & a(f'''(1+g'^2) + 2f'f''g'')\sqrt{W} + a(f''(1+g'^2) + g''(1+f'^2))\frac{f'f''}{\sqrt{W}} \\ & + bf'''g'' = 4f'f''W. \\ & a(2f''g'g'' + g'''(1+f'^2))\sqrt{W} + a(f''(1+g'^2) + g''(1+f'^2))\frac{g'g''}{\sqrt{W}} \\ & + bf''g''' = 4g'g''W. \end{aligned}$$

From both equations, we obtain the value of  $W$  on the right hand sides and we equal both expressions, deducing

$$(12) \quad a \left( \frac{f'''}{f'f''}(1+g'^2) + 2g'' - 2f'' - \frac{g'''}{g'g''}(1+f'^2) \right) \sqrt{W} = b \left( f'' \frac{g'''}{g'g''} - g'' \frac{f'''}{f'f''} \right).$$

If we write (11) and (12) as  $P_1\sqrt{W} = Q_1$  and  $P_2\sqrt{W} = Q_2$ , respectively, we obtain  $P_1Q_2 - P_2Q_1 = 0$ . After some manipulations, this identity writes as

$$(f'f''^2g''' - f'''g'g''^2)(2f'f''g'g''(f'' - g'') + f'f''(1+f'^2)g''' - f'''g'g''(1+g'^2)) = 0,$$

that is,  $P_2Q_2 = 0$ . We discuss by cases:

- (1) Case  $P_2 = 0$  and  $Q_2 \neq 0$ . Then (12) implies  $a = 0$ , a contradiction.
- (2) Case  $P_2 \neq 0$  and  $Q_2 = 0$ . Then (12) implies  $b = 0$ , a contradiction.
- (3) Case  $P_2 = Q_2 = 0$ . These two equations write as

$$(13) \quad \frac{f'''}{f'f''^2} = \frac{g'''}{g'g''^2}$$

$$(14) \quad 2(f'' - g'') + \frac{g'''}{g'g''}(1+f'^2) - \frac{f'''}{f'f''}(1+g'^2) = 0.$$

Equation (13) implies the existence of  $\lambda \in \mathbb{R}$  such that

$$(15) \quad \frac{f'''}{f'f''^2} = \frac{g'''}{g'g''^2} = 2\lambda$$

and thus

$$\frac{f'''}{f'f''} = 2\lambda f'', \quad \frac{g'''}{g'g''} = 2\lambda g''.$$

Substituting the above in (14), we get

$$2(f'' - g'') + 2\lambda(1+f'^2)g'' - 2\lambda(1+g'^2)f'' = 0,$$

or

$$(16) \quad f'' - g'' + \lambda g'' - \lambda f'' = \lambda f''g'^2 - \lambda g''f'^2.$$

If  $\lambda \neq 0$ , differentiating this equation with respect to  $x$  and then with respect to  $y$ , we deduce

$$f'f''g''' = g'g''f'''.$$

As we suppose that  $f'', g'' \neq 0$ , we conclude that

$$\frac{f'''}{f'f''} = \frac{g'''}{g'g''} = \mu$$

for some constant  $\mu \in \mathbb{R}$ . Substituting in (15) we deduce that  $f'', g''$  are both constant functions, so (15) yields to  $\lambda$  being zero, a contradiction.

Therefore,  $\lambda = 0$  in (15). Equation (16) says now that  $f'' = g'' = m$ , for some real number  $m \neq 0$ . Then (9) writes as

$$am(2 + f'^2 + g'^2) = W^{\frac{3}{2}} - bm^2W^{-\frac{1}{2}}.$$

Differentiating with respect to  $x$  and simplifying by  $f'f''$ , we get

$$2am = 3W^{\frac{1}{2}} + bm^2W^{-\frac{3}{2}},$$

which implies that  $W$  is constant and this would say that  $f'' = g'' = 0$ , a contradiction.

### 3. THE LORENTZIAN CASE

We consider the Lorentz-Minkowski space  $\mathbb{L}^3$ , that is, the real vector space  $\mathbb{R}^3$  endowed with the metric  $(dx)^2 + (dy)^2 - (dz)^2$  where  $(x, y, z)$  are the canonical coordinates. A surface  $S$  immersed in  $\mathbb{L}^3$  is said non degenerate if the induced metric on  $S$  is non degenerated. The induced metric on  $S$  can only be of two types: positive definite and the surface is called spacelike, or a Lorentzian metric, and the surface is called timelike. For both types of surfaces, it is defined the mean curvature  $H$  and the Gauss curvature  $K$  and we say again that the surface is of linear Weingarten type if there exists a linear relation between  $H$  and  $K$  as in (1).

Similarly, in Lorentzian setting we can extend the concept of translation surface. A surface  $S$  in  $\mathbb{L}^3$  is again locally a graph on one of the coordinate planes, since this property is not metric but because  $S$  is immersed. Thus a translation surface in  $\mathbb{L}^3$  is a surface that writes locally as the graph of a function which is the sum of two real functions. However, in  $\mathbb{L}^3$  we can say a bit more. If  $S$  is spacelike, then  $S$  is a graph on the  $xy$ -plane and if  $S$  is a timelike surface, then  $S$  is a graph on the  $xz$ -plane or on the  $yz$ -plane [14]. Therefore, if  $S$  is a translation surface in  $\mathbb{L}^3$ , we may suppose that:

- (1) If  $S$  is spacelike, then  $S$  writes locally as  $z = f(x) + g(y)$ .
- (2) If  $S$  is timelike, then  $S$  writes locally as  $y = f(x) + g(z)$  or as  $x = f(y) + g(z)$ .

In [2], Theorem A was extended to non-degenerate surfaces of  $\mathbb{L}^3$ , obtaining a similar result. Again, in this classification, the only translation surfaces of linear Weingarten type appear with trivial choices of  $a$  and  $b$  and the surfaces have constant  $H$  or constant  $K$ . Similarly, we extend Theorem 1 as follows:

**Theorem 2.** *A translation non-degenerate surface in Lorentz-Minkowski space  $\mathbb{L}^3$  of linear Weingarten type is a surface with constant Gauss curvature  $K$  or constant mean curvature  $H$ .*

Translations surfaces in  $\mathbb{L}^3$  with constant mean curvature or constant Gauss curvature were classified in [8] and they are a plane, a Scherk's minimal surface or a generalized cylinder.

*Proof.* The proof of Th. 2 is similar as Th. 1 and we only sketch the differences. Moreover, we will carry jointly the cases that the surface  $S$  is spacelike or timelike. Again, we suppose by contradiction that  $a, b \neq 0$  in (1). The expressions of  $H$  and  $K$  in local coordinates are

$$H = \epsilon \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}, \quad K = \epsilon \frac{eg - f^2}{EG - F^2},$$

where  $\epsilon = -1$  if  $S$  is spacelike and  $\epsilon = 1$  if  $S$  is timelike ([10, 14]).

Suppose that  $S$  writes as  $z = f(x) + g(y)$  if  $S$  is spacelike or  $y = f(x) + g(z)$  if  $S$  is timelike. Then

$$H = \epsilon \frac{-\epsilon f''(1 - g'^2) + g''(1 + \epsilon f'^2)}{2((1 + \epsilon f'^2 - g'^2))^{\frac{3}{2}}}, \quad K = -\frac{f''g''}{(1 + \epsilon f'^2 - g'^2)^2},$$

with  $W = 1 + \epsilon f'^2 - g'^2 > 0$ . Let

$$F = \frac{f''}{1 + \epsilon f'^2}, \quad G = \epsilon \frac{g''}{-1 + g'^2}.$$

If  $c = 0$  in (1), then (7) is the same, obtaining (8). This implies that  $W$  is constant, a contradiction.

If  $c \neq 0$ , then we assume after a change of constants  $a$  and  $b$  that  $c = 1$ . Now the linear Weingarten condition (1) expresses as

$$(17) \quad a(F + G)\sqrt{W} + bFG = \varepsilon \frac{W^2}{(1 + \epsilon f'^2)(-1 + g'^2)}.$$

Now (11) and (12) write, respectively, as

$$\begin{aligned} & a \left( \frac{F'}{f'f''} + \frac{2(F + G)}{\varepsilon + f'^2} + \varepsilon \frac{G'}{g'g''} + \varepsilon \frac{2(F + G)}{-1 + g'^2} \right) \sqrt{W} \\ & + b \left( \frac{F'G}{f'f''} + \frac{2FG}{\varepsilon + f'^2} + \varepsilon \frac{FG'}{g'g''} + \varepsilon \frac{2FG}{-1 + g'^2} \right) = 0 \\ & a \left( \frac{f'''}{f'f''}(-1 + g'^2) + 2g'' + 2\varepsilon f'' + \varepsilon(\varepsilon + f'^2) \frac{g'''}{g'g''} \right) \sqrt{W} \\ & + \varepsilon b \left( f'' \frac{g'''}{g'g''} + \varepsilon g'' \frac{f'''}{f'f''} \right) = 0. \end{aligned}$$

We deduce

$$\begin{aligned} & (f'f''^2g''' + \varepsilon f'''g'g''^2) (2f'f''g'g''(f'' + \varepsilon g'') + f'f''(f'^2 + \varepsilon)g''') \\ & + \varepsilon f'''g'g''(g'^2 - 1) = 0 \end{aligned}$$

and now the discussion by cases is similar as it was done in the Euclidean case, obtaining that  $W$  is constant, a contradiction.  $\square$

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